Surface Design using Locally Interpolating Subdivision Schemes

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Non-uniform multivariate subdivision schemes are constructed, which generate limit functions interpolating some of the initial control points. Our schemes differ from the known interpolatory subdivision schemes, in that only some of the original control points are interpolated, and not the control points in every level. These new schemes are combinations of a non-interpolatory schemes with different local schemes near some of the original control points. They generate smooth surfaces interpolating given points, using stencils of small support. Next, it is shown how to modify known subdivision schemes so that the limit surfaces generated by them interpolate given normal vectors at given interpolation points. © 2000 Academic Press

Key Words: subdivision schemes; finitely non-uniform subdivision schemes; interpolation; normal vector interpolation.

1. INTRODUCTION

Subdivision schemes have been studied extensively, as a tool for curve/surface design [1, 3, 5]

Interpolatory subdivision schemes are constructed and analyzed in [4, 6, 9] and are of great interest, since every control point generated by such a subdivision process, lies on the limit surface, which is very intuitive to the designer. However, the basis functions generated by applying interpolatory subdivision schemes to the Kronecker initial data, exhibit bad behavior, such as an infinite number of inflection points! Moreover, experience shows that the stencils of interpolatory subdivision schemes, have large support with respect to the smoothness of their limit functions.

In this paper, we construct new non-uniform subdivision schemes, that generate limit surfaces interpolating some of the original control points only. This is done by applying a non-interpolatory subdivision scheme almost everywhere, except for areas around special control points, where

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another local stencil is used. Our schemes use simpler stencils of smaller support and increased smoothness, in comparison with interpolatory subdivision schemes. For the simplicity of the analysis, we construct schemes that interpolate only the control point at the origin. The same local schemes that work near the origin, can be used in the neighborhood of any initial control point, providing the interpolation property to initial control points.

Section 2 gives the notations and background on subdivision schemes. Section 3 defines *finitely non-uniform subdivision schemes* and presents a construction of such schemes. An example for this construction is given in Section 4. Section 5 provides a sufficient condition for the limit surfaces to be C^m for a given finitely non-uniform scheme. In Section 6 we use this sufficient condition to construct families of C^2 finitely non-uniform schemes. In Section 7 we present a method for the construction of finitely non-uniform schemes that can also interpolate given normal vectors. Section 8 depicts some surfaces that were generated by our schemes.

2. SUBDIVISION SCHEMES

Given $X \subset \mathbb{Z}^s$, let $l(X)$ denote all the functions $P: X \to \mathbb{R}$, let $l_\infty(X)$ denote the Banach space of all the functions $P: X \to \mathbb{R}$ such that $\|P\|_\infty < \infty$, where $\|P\|_\infty$ is the supremum of $|P|$ on X. Let $l_0(X) \subset l_\infty(X)$ denote the space of all the functions $P \in l_\infty(X)$ with finite support.

A subdivision operator is a linear operator $S: l(\mathbb{Z}^s) \to l(\mathbb{Z}^s)$ which is based on a mask $a \in l_0(\mathbb{Z}^s)$ and is defined by

$$(SP)(\alpha) = \sum_{\beta \in \mathbb{Z}^s} a(\alpha - 2\beta) P(\beta), \qquad \forall \alpha \in \mathbb{Z}^s. \tag{1}$$

S is also referred to as a *subdivision scheme*, when it is repeatedly applied to data from $l(\mathbb{Z}^s)$. S is called an interpolatory scheme, if

$$(SP)(2\alpha) = P(\alpha), \qquad \forall \alpha \in \mathbb{Z}^s, \quad \forall P \in l(\mathbb{Z}^s). \tag{2}$$

A subdivision scheme S is termed *uniformly convergent*, if for every $P \in l_0(\mathbb{Z}^s)$, there exists a compactly supported function $F \in C(\mathbb{R}^s)$ (called the limit function) such that

$$\lim_{n \to \infty} \|S^n P - F(2^{-n} \cdot)\|_{\infty, \mathbb{Z}^s} = 0. \tag{3}$$

We denote $S^\infty P = F$. It follows from (3) that

$$S^\infty SP(\cdot) = S^\infty P(\cdot 2). \tag{4}$$

Specifically, we denote $\Phi = S^\infty \delta_0$ where $\delta_0(\alpha)$ attains the value 1 for $\alpha = 0$, and 0 otherwise. Φ is called the S-refinable function. The limit function $S^\infty P$ can be expressed as a sum of integer translates of Φ

$$S^\infty P = \sum_{\alpha \in \mathbb{Z}^s} P(\alpha)\, \Phi(\,\cdot\, - \alpha) = \sum_{\alpha \in \mathbb{Z}^s} S^k P(\alpha)\, \Phi(2^k \cdot - \alpha), \qquad k \geqslant 0. \quad (5)$$

We say that S reproduces the polynomial $p \in \pi^m$, if

$$S(p_{|\mathbb{Z}^s})(2\,\cdot\,) = p_{|\mathbb{Z}^s}, \tag{6}$$

where π^m is the space of polynomials of degree $\leqslant m$. Throughout the paper we use the standard multi-index notations for $\mathbb{Z}^s$,

$$\alpha = (\alpha_1, ..., \alpha_s) \in \mathbb{Z}^s, \quad \alpha \geqslant 0 \text{ if } \alpha_1, ..., \alpha_s \geqslant 0, \quad |\alpha| = \sum_{i=1}^s \alpha_i,\, x^\alpha = x_1^{\alpha_1} \cdot \,\cdots\, \cdot x_s^{\alpha_s},$$

$$\alpha! = \alpha_1! \cdot \,\cdots\, \cdot \alpha_s!, \qquad D^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_s} x_s}.$$

3. A METHOD FOR CONSTRUCTING FINITELY NON-UNIFORM INTERPOLATING SCHEMES

In this section, we show that for a large family of subdivision schemes S, it is easy to construct a non-uniform scheme $\tilde{S}$ that coincides with S away from the origin, has the interpolatory property at the origin, and generates limit functions from the same shift invariant space of functions generated by S.

DEFINITION. A linear operator $\tilde{S}: l(\mathbb{Z}^s) \to l(\mathbb{Z}^s)$ is called a *finitely non-uniform subdivision operator* if there exists a subdivision operator S and a finite set $T \subset \mathbb{Z}^s$ such that the support of $(\tilde{S} - S)\, P$ is contained in T for all $P \in l(\mathbb{Z}^s)$, and such that $P_{|T} = 0 \Rightarrow \tilde{S} P_{|T} = 0$ for all P.

In other words, a finitely non-uniform subdivision operator differs from a uniform subdivision operator only on a finite set T, and the values of $\tilde{S}P$ in T depend only on the values of P in T. We also refer to $\tilde{S}$ as a finitely non-uniform subdivision scheme.

Given a subdivision scheme S with a basis function Φ satisfying $\Phi(0) \neq 0$, we define an operator $L: l(\mathbb{Z}^s) \to l(\mathbb{Z}^s)$ by

$$LP(\alpha) = \begin{cases} P(\alpha), & \alpha \neq 0, \\ S^\infty P(0), & \alpha = 0. \end{cases} \tag{7}$$

L is invertible since $\Phi(0) \neq 0$:

$$L^{-1}Q(\alpha) = \begin{cases} Q(\alpha), & \alpha \neq 0, \\ (Q(0) - \sum_{\beta \neq 0} Q(\beta)\,\Phi(-\beta))\,\Phi(0)^{-1}, & \alpha = 0. \end{cases} \quad (8)$$

Let $\tilde{S} = LSL^{-1}$. We will now show that the limit functions generated by $\tilde{S}$ belong to the same shift-invariant space of functions generated by S. Let $P^0 = P$, $Q^0 = L^{-1}P$, and define

$$P^{n+1} = \tilde{S}P^n, \qquad n > 0, \quad (9)$$

$$Q^{n+1} = SQ^n, \qquad n > 0. \quad (10)$$

Then

$$P^n = LQ^n, \qquad n \geqslant 0. \quad (11)$$

Thus P^n coincides with Q^n everywhere except for the origin. Moreover, we have $P^n(0) = P(0)$ for all $n \geqslant 0$, since (4) yields

$$P^{n+1}(0) = LQ^{n+1}(0) = S^\infty Q^{n+1}(0) = S^\infty SQ^n(0)$$
$$= S^\infty Q^n(0) = LQ^n(0) = P^n(0). \quad (12)$$

Let $F = S^\infty Q^0$, then

$$\lim_{n \to \infty} \|Q^n - F(2^{-n}\cdot)\|_{\infty, \mathbb{Z}^s} = 0. \quad (13)$$

Because $F(0) = P(0)$, and $P^n(\alpha) = Q^n(\alpha)$ for $\alpha \neq 0$, we get from (13) that

$$\lim_{n \to \infty} \|P^n - F(2^{-n}\cdot)\|_{\infty, \mathbb{Z}^s} = 0, \quad (14)$$

showing that the subdivision scheme $\tilde{S}$ is uniformly convergent. The smoothness of its limit functions, is the same as the smoothness of the limit functions generated by S.

The structure of $\tilde{S}$ is simple: It operates exactly like S everywhere, except on a finite neighborhood of the origin, whose size is determined by the support of Φ. In particular (12) means that $\tilde{S}$ does not change the value of its operand at the origin, which provides the interpolation property.

4. A FINITELY NON-UNIFORM C^2 INTERPOLATING SCHEME FROM A BOX-SPLINE SCHEME

A 3 directional box-spline with multiplicities 2 is generated by the subdivision scheme S defined by the following mask supported on $\{-2, ..., 2\}^2$

$$a = \begin{bmatrix} 0 & 0 & 116 & 18 & 116 \\ 0 & 18 & 38 & 38 & 18 \\ 116 & 38 & 58 & 38 & 116 \\ 18 & 38 & 38 & 18 & 0 \\ 116 & 18 & 116 & 0 & 0 \end{bmatrix}. \tag{15}$$

It can be shown that S generates C^2 limit surfaces [2]. The extension of this subdivision scheme to general triangulations is also known as Loop's scheme [8]. Using the method in section 3, we construct from the scheme S a finitely non-uniform interpolating C^2 scheme. It is easy to see that

$$LP(0) = LP(0, 0) = S^\infty P(0) = 12P(0, 0) + 112P(1, 0) + 112P(1, 1)$$
$$+ 112P(0, 1) + 112P(-1, 0) + 112P(-1, -1) + 112P(0, -1). \tag{16}$$

Therefore

$$L^{-1}P(0) = 2P(0, 0) - 16P(1, 0) - 16P(1, 1)$$
$$- 16P(0, 1) - 16P(-1, 0) - 16P(-1, -1) - 16P(0, -1). \tag{17}$$

We represent L and L^{-1} at the origin by the two stencils shown in Fig. 1.
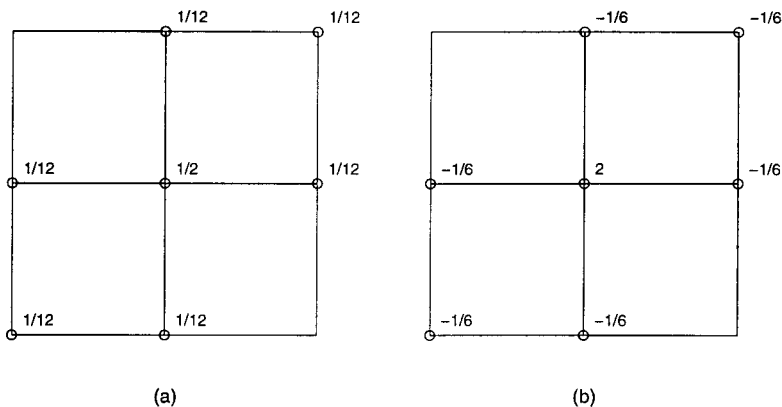

FIG. 1. (a) A stencil representing L at the origin, (b) a stencil representing L^{-1} at the origin.

An implementation of the scheme $\tilde{S} = LSL^{-1}$ can be done by first applying L^{-1}, then S, and then set the value at the origin to its original value, which corresponds to an application of L. Both L^{-1} and S consist of "one-ring" stencils, i.e. new controls points depend on first-order neighbors only. However, $\tilde{S}$ is a "two-ring" scheme, i.e. a new control point depends on second-order neighbors. We aim to construct "one-ring" stencils knowing that the locality of a scheme affects significantly the complexity of the computations involved in it.

5. CONDITIONS FOR CONVERGENCE AND SMOOTHNESS OF THE LIMIT FUNCTIONS

Given a finitely non-uniform subdivision scheme $\tilde{S}$, we formulate conditions for $\tilde{S}$ to converge and for its limit functions to be C^m. First, here are some notations and definitions for finitely non-uniform schemes.

Let $T \subset \mathbb{Z}^s$ denote a finite set that contains the origin. Let M_T denote a square matrix representing a linear operator $M_T: \mathbb{R}^T \to \mathbb{R}^T$. Let S denote a uniform subdivision scheme that is known to generate limit functions in $C^m(\mathbb{R}^s)$. A finitely non-uniform operator $\tilde{S}$ has the form

$$\tilde{S}P(\alpha) = \begin{cases} M_T P_{|T}(\alpha) & \alpha \in T, \\ SP(\alpha), & \alpha \notin T, \end{cases} \quad \forall P. \tag{18}$$

We call the matrix M_T a *refinement matrix*, as it represents the operation of $\tilde{S}$ on the *refinement set T*. Let $P^0 = P \in l(\mathbb{Z}^s)$ denote initial control points, and define the subdivision scheme by

$$P^{n+1} = \tilde{S}P^n, \quad n \geqslant 0. \tag{19}$$

We say that the scheme $\tilde{S}$ converges uniformly to a limit function F over the open set $D \subset \mathbb{R}^s$ if

$$\lim_{n \to \infty} \| P^n - F(2^{-n} \cdot) \|_{\infty, \mathbb{Z}^s \cap 2^n D} = 0. \tag{20}$$

We denote that limit function $\tilde{S}^\infty P$. For every $x \in \mathbb{R}^s \backslash \{0\}$, the scheme $\tilde{S}$ coincides with S in the neighborhood of $2^n x$ when n is big enough, therefore the limit function is well defined everywhere except maybe at the origin. In particular, far away from the origin, only the scheme S operates. Therefore, there exists an open and bounded set $\Theta \subset \mathbb{R}^s$ that contains the origin, with the property that

$$x \in \mathbb{R}^s \backslash \Theta \Rightarrow \tilde{S}^\infty P(x) = S^\infty P(x) = \sum_{\alpha \in \mathbb{Z}^s \backslash \{0\}} P(\alpha)\, \Phi(x - \alpha), \quad \forall P. \tag{21}$$

Due to the locality of S, T can be enlarged (and M_T modified accordingly) without changing the operator $\tilde{S}$, so that (5.1) remains valid, and so that the values of $S^\infty P$ inside $2\Theta\backslash\Theta$ depend only on values of P in $T\backslash\{0\}$. Therefore, together with (5.4) this yields

$$x \in 2\Theta\backslash\Theta \Rightarrow \tilde{S}^\infty P(x) = \sum_{\alpha \in T\backslash\{0\}} P(\alpha)\,\Phi(x-\alpha), \qquad \forall P, \quad \forall n \geqslant 0, \quad (22)$$

Observing that

$$\tilde{S}^\infty P = \tilde{S}^\infty P^n(2^n \cdot), \qquad \forall P, \quad (23)$$

we get that

$$x \in 2^{-n+1}\Theta\backslash 2^{-n}\Theta \Rightarrow \tilde{S}^\infty P(x) = \sum_{\alpha \in T\backslash\{0\}} P^n(\alpha)\,\Phi(2^n x - \alpha). \quad (24)$$

For every P, and for every $n \geqslant 0$. A sufficient condition for the convergence of P^n to a continuous limit function is provided by

THEOREM 1. *If M_T has a simple eigenvalue* 1 *with the corresponding eigenvector* $(1, ..., 1)$, *and all the other eigenvalues of modulus less than* 1, *than $\tilde{S}^n P$ converges to a continuous limit function, for all P. The value of the limit function at the origin is the Euclidean inner product* $(w, P_{|T})$ *where w is the eigenvector of M_T^t with eigenvalue* 1, *normalized such that its coordinates sum up to* 1.

Proof. From the assumption that S is uniformly convergent, we know that it reproduces constants [1]. By the conditions of the theorem, we deduce that $\tilde{S}$ also reproduces constants. It is known that for any $0 < \rho < 1$ that has modulus greater than the second eigenvalue of M_T,

$$M_T^n P_{|T} = (w, P_{|T})(1, ..., 1) + o(\rho^n), \qquad n \to \infty. \quad (25)$$

Let

$$Q(\alpha) = P(\alpha) - (w, P_{|T}), \qquad \alpha \in \mathbb{Z}^s. \quad (26)$$

We already know that $\tilde{S}^n(P - Q)$ converges to the constant function with constant value $(w, P_{|T})$. We also know from (25) and (26) that

$$M_T^n Q_{|T} = o(\rho^n), \qquad n \to \infty. \quad (27)$$

We only have to show that $\tilde{S}^n Q$ converge to a continuous function with value 0 at the origin. From (24) and (27) we get that

$$x \in 2^{-n+1}\Theta\backslash 2^{-n}\Theta \Rightarrow \tilde{S}^\infty Q(x) \leqslant C\rho^n, \quad (28)$$

therefore

$$\tilde{S}^\infty Q(x) = o(\|x\|^\gamma), \qquad \gamma = -\frac{\log(\rho)}{\log(2)} > 0. \quad \blacksquare \qquad (29)$$

Next, we refine the analysis of the eigenproperties in Theorem 1 to guarantee the smoothness of the limit functions $\tilde{S}^\infty P$. In particular the scheme $\tilde{S}$ is now required to have polynomial reproduction properties. While C^m schemes always send the space of polynomials up to degree m into itself, non-interpolatory C^m schemes do not generally reproduce all the polynomials of degree m, in the sense defined by (6). However, we make use of the fact that some interesting schemes reproduce a subspace of polynomials of degree $\leqslant m$ (see Section 6).

THEOREM 2. *Assume that every eigenvalue of M_T that has modulus $\geqslant 2^{-m}$ is equal 2^{-r} for some integer $0 \leqslant r \leqslant m$, and that its algebraic multiplicity is the same as its geometric multiplicity. If every eigenvector that corresponds to the eigenvalue 2^{-r} with $0 \leqslant r \leqslant m$ has the form $p_{|T}$ where p is a homogeneous polynomial of degree r that is reproduced by S, then $\tilde{S}$ generates C^m limit functions for every initial data.*

Proof. Let $\lambda_1, ..., \lambda_k$ denote the eigenvalues of M_T with modulus $\geqslant 2^{-m}$, and let $v_1, ..., v_k$ denote the corresponding eigenvectors. From the conditions of the theorem, there exist $p_1, ..., p_k \in \pi_0^m$ such that

$$v_i = p_{i_{|T}}, \qquad i = 1, ..., k, \qquad (30)$$

where π_0^m denotes the set of homogeneous polynomials of degree $\leqslant m$. For each initial data P, $P_{|T}$ can be expanded in terms of the generalized eigenvalues of M_T. If we take $a_1, ..., a_k$ to be the coefficients of $v_1, ..., v_k$ in that expansion, then there exists $0 < \rho < 2^{-m}$ such that

$$M_T^n P_{|T} = \sum_{i=1}^{k} a_i \lambda_i^n v_i + o(\rho^n), \qquad n > 0. \qquad (31)$$

Define Q by

$$Q = P - \left(\sum_{i=1}^{k} a_i p_i \right)_{|\mathbb{Z}^s}. \qquad (32)$$

We know from the polynomial reproduction condition of the theorem that

$$\tilde{S}^{\infty}(P - Q) = \sum_{i=1}^{k} a_i p_i \in \pi^m(\mathbb{R}^s). \tag{33}$$

Since by (24), $S^{\infty}P \in C^m(\mathbb{R}^s \backslash \{0\})$ for all P, we only need to show that

$$D^j \tilde{S}^{\infty} Q(x) \to 0 \qquad \text{as} \quad x \to 0, \qquad \forall P, \quad \forall 0 \leqslant |j| \leqslant m. \tag{34}$$

From (31) and (32), we get

$$M_T^n Q_{|T} = o(\rho^n), \qquad n > 0, \tag{35}$$

and from (24) we have

$$x \in 2^{-n+1}\Theta \backslash 2^{-n}\Theta \Rightarrow D^j \tilde{S}^{\infty} Q(x)$$

$$= \sum_{\alpha \in T \backslash \{0\}} M_T^n Q_{|T}(\alpha) \, 2^{n|j|} D^j \Phi(2^n x - \alpha), \qquad \forall P, \quad \forall 0 \leqslant |j| \leqslant m \tag{36}$$

therefore

$$D^j \tilde{S}^{\infty} Q(x) = o(\|x\|^{\gamma}), \tag{37}$$

for all P, and $0 \leqslant |j| \leqslant m$, where $\gamma = -\log(2^{|j|}\rho) \log(2) > 0$. ∎

The following corollary can be used to calculate the first order partial derivatives of the limit function at the origin, and specifically to establish the regularity of limit surfaces in case $s = 2$.

COROLLARY 3. *If the conditions of Theorem* 2 *are fulfilled with* $m = 1$, *and* M_T *has the eigenvalue* $\lambda = 12$ *with exact algebraic and geometric multiplicity* s, *then there exist linearly independent vectors* $w_1, ..., w_s \in \mathbb{R}^T$ *such that*

$$D^{e_i} \tilde{S}^{\infty} P(0) = (w_i, P_{|T}), \qquad \forall P, \tag{38}$$

where $\{e_i\}_{i=1,...,s}$ *is the standard basis of* $\mathbb{R}^s$. *The vectors* w_i *are in the subspace* $\{v \in \mathbb{R}^T \mid M_T^t v = 12v\}$, *and they are a biorthogonal basis to the basis* $\{(\cdot)_{|T}^{e_i}\}_{i=1,...,s}$.

Proof. Following the lines of the proof of Theorem 2, we know from the conditions of the theorem that there exist linearly independent $w_1, ..., w_s$ (which are eigenvectors of M_T^t corresponding to the eigenvalue 12) and $0 < \rho < 12$ such that

$$M_T^n P_{|T} = \sum_{i=1}^{s} a_i (12)^n p_{i_{|T}} + p_{0_{|T}} + O(\rho^n), \tag{39}$$

where $a_i = (w_i, P_{|T})$ and $p_i(\cdot) = (\cdot)^{e_i}$, $i = 1, ..., s$ denote linear monomials, and $p_0 \in \pi^0$. Define Q by

$$Q = P - \left(\sum_{i=1}^{s} a_i p_i \right)_{|\mathbb{Z}^s}. \tag{40}$$

Then using (33) and (34) we get

$$D^{e_i} \tilde{S}^\infty P(0) = (w_i, P_{|T}), \qquad i = 1, ..., s. \quad \blacksquare \tag{41}$$

In Section 7 we make use of schemes that satisfy the condition of Corollary 4, to prescribe normal vectors of the limit surfaces.

COROLLARY 4. *If the conditions of Theorem* 2 *are fulfilled with* $m = 1$, *and* M_T *does not have* 12 *as an eigenvalue, then the gradient of* $\tilde{S}^\infty P$ *at the origin is zero, for all* P.

Theorem 2 can be used to construct schemes whose limit functions are in C^m. Another class of functions which are also of interest, contains the functions that are in C^{m-1} with bounded derivatives of order m. For example, in Computer Aided Machining (CAM) surfaces are required to be smooth, or at least piecewise smooth, but the curvature of the surfaces does not have to be continuous. In order to enable machining, the surfaces must have only *bounded* curvature. A weaker version of Theorem 2 provides the corresponding sufficient condition:

THEOREM 5. *Assume that every eigenvalue of* M_T *that has modulus* $\geqslant 2^{-m}$ *is equal* 2^{-r} *for some integer* $0 \leqslant r \leqslant m$, *and that its algebraic multiplicity is the same as its geometric multiplicity. If every eigenvalue corresponding to the eigenvector* 2^{-r}, *with* $0 \leqslant r < m$ *has the form* $p_{|T}$ *where* p *is a homogeneous polynomial of degree* r *that is reproduced by* S, *then* $\tilde{S}$ *generates* C^{m-1} *limit functions, which are* C^m *everywhere except at the origin, and whose derivatives of order m are bounded in the neighborhood of the origin for every initial data.*

The above conditions are weaker than the conditions of Theorem 2, since they allow eigenvalues of modulus 2^{-m} other than $\lambda = 2^{-m}$, and corresponding eigenvectors that are not polynomials restricted to T.

Proof. Following the proof of Theorem 2, we only have to prove that the mth order derivatives of the function $\tilde{S}^{\infty}P$ are bounded in the neighborhood of the origin, when $P_{|T}$ is an eigenvector of M_T with eigenvalue of modulus 2^{-m}. Let $j = (j_1, ..., j_s)$ with $|j| = m$. From (24) we have

$$x \in 2^{-n+1}\Theta \setminus 2^{-n}\Theta \Rightarrow D^j \tilde{S}^{\infty}P(x)$$
$$= \sum_{\alpha \in T \setminus \{0\}} M_T^n P_{|T}(\alpha)\, 2^{mn}\, D^j\Phi(2^n x - \alpha). \tag{42}$$

But

$$|2^{mn} M_T^n P_{|T}(\alpha)| = |P(\alpha)|, \qquad \forall \alpha \in T, \tag{43}$$

therefore,

$$x \in 2^{-n+1}\Theta \setminus 2^{-n}\Theta \Rightarrow |D^j \tilde{S}^{\infty}P(x)| \leqslant \sum_{\alpha \in T \setminus \{0\}} |P(\alpha)|\, D^j\Phi(2^n x - \alpha). \tag{44}$$

Finally we get

$$\|D^j \tilde{S}^{\infty}P\|_{\infty, 2\Theta \setminus \{0\}} \leqslant \|P\|_{1, T} \sup_{|j| = m} \|D^j\Phi\|_{\infty}. \quad \blacksquare \tag{45}$$

6. EXAMPLES: FINITELY NON-UNIFORM C^2 INTERPOLATING SCHEMES.

In Section 4, we have shown a construction of a C^2 interpolating scheme which coincides with a Box-Spline scheme S away from the origin. However, as pointed out in the last paragraph of that section, this results in a two-ring scheme. Using the sufficient condition given by Theorem 2, we construct a one-ring finitely non-uniform interpolating scheme which is C^2.

We seek the values of the weights that correspond to a stencil near the origin, and set up the equations for them to satisfy the smoothness conditions. Figure 2 depicts the stencil that is used to calculate $\tilde{S}P(1, 0)$. The choice of coefficients is made such that three-directional symmetry is kept (with respect to the directions $(0, 1)$, $(1, 0)$, $(1, 1)$). Our goal is to find values of a, b, c, d, e that guarantee C^2 limit surfaces.
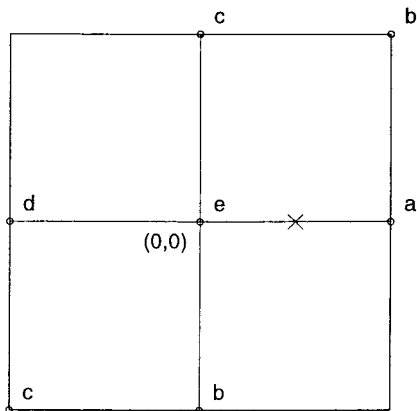
FIG. 2. A one-ring stencil.

The stencils for calculation of $\tilde{S}P$ at the points $(1, 1)$, $(0, 1)$, $(-1, 0)$, $(-1, -1)$, $(0, -1)$ are derived by requiring the finitely non-uniform scheme to exhibit the same 3-directional symmetry as the original uniform scheme. We already know that S reproduces the following homogeneous polynomials in $\mathbb{Z}^2$: 1, x, y, $x^2 - 2xy$, $y^2 - 2xy$, therefore, we require that (a) $\tilde{S}$ also reproduces these polynomials, and (b) all eigenvalues of the refinement matrix M_T, except those that correspond to reproduced polynomials, have modulus less than 14.

Requirement (a) reduces to the following set of equations

$$
\begin{aligned}
a + 2b + 2c + d + e &= 1, \\
a - c &= 38, \\
b - d &= 18.
\end{aligned} \tag{46}
$$

Requirement (b) reduces to the inequalities

$$
\begin{aligned}
|a + 2b + 2c + d| &< 14, \\
|a - 2b + 2c - d| &< 14.
\end{aligned} \tag{47}
$$

The general solution of (46)–(47) can be put in the form

$$
\begin{aligned}
a &= \alpha + \beta + 14, \\
b &= \alpha - \beta + 124, \\
c &= \alpha + \beta - 18, \\
d &= \alpha - \beta - 112, \\
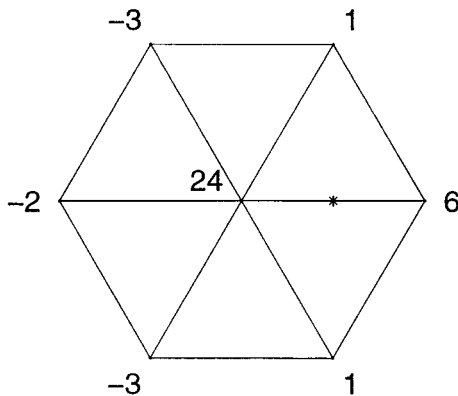e &= 1 - 6\alpha,
\end{aligned} \tag{48}
$$

FIG. 3. A stencil near interpolatory vertices of a regular triangulation.

with the necessary and sufficient condition for C^2 smoothness:

$$|\alpha|, |\beta| < 124. \tag{149}$$

Due to the three-directional symmetry, this scheme can be used near regular vertices of a triangulation (i.e., vertices with valency 6). The stencil with $\alpha = \beta = 0$ is depicted in Fig. 3.

We can use the two degrees of freedom α, β to improve the shape of the limit surfaces. It seems good practice to seek such values α, β that minimize a certain *fairness measure* of a limit surface of the subdivision process, starting with certain *initial control points*. There are many possible choices of fairness criteria and of the initial control points. We chose a fairness measure that considers third order derivatives of the function

$$E(f) = \int f_{xxx}^2 + 3f_{xxy}^2 + 3f_{xyy}^2 + f_{yyy}^2 . \tag{50}$$

Let

$$\delta_\eta(\mu) = \begin{cases} 1, & \mu = \eta, \\ 0, & \text{otherwise.} \end{cases}$$

The only values of η for which $\tilde{S}^\infty \delta_\eta$ depends on α and β are first order neighbors of the origin, and considering the three-directional symmetry, the
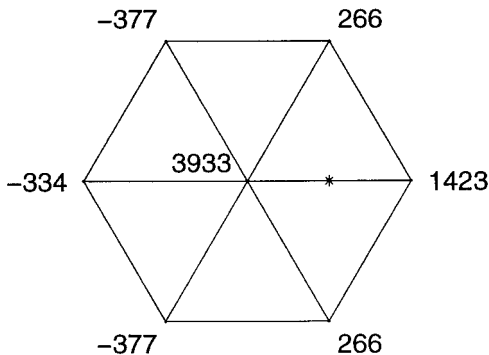
FIG. 4. A stencil calculated by minimizing a fairness measure. The coefficients sum up to 4800.

only distinct values of $E(\tilde{S}^\infty \delta_\eta)$ among those values of η, are the ones with $\eta = (0, 0), (1, 1), (1, 0)$.

Therefore, we focus on three sets of initial control points: $\delta_{(0, 0)}$, $\delta_{(1, 0)}$, $\delta_{(1, 1)}$. We search for α, β that minimize the quantity

$$E(\tilde{S}^\infty \delta_{(0, 0)}) + E(\tilde{S}^\infty \delta_{(1, 0)}) + E(\tilde{S}^\infty \delta_{(1, 1)}). \tag{51}$$

We use a finite difference approximation for $E(\tilde{S}^\infty P)$, taken from the values of $\tilde{S}^4 P$. Finally, we round the values of α, β to rational numbers which are useful for exact arithmetic. Figure 4 depicts the resulting stencil.

In a similar way, we construct a scheme that is suitable for subdivision of quadrilateral meshes near a regular vertex (i.e., with valency 4). As the scheme S away from the origin, we use the Catmull–Clark scheme, which
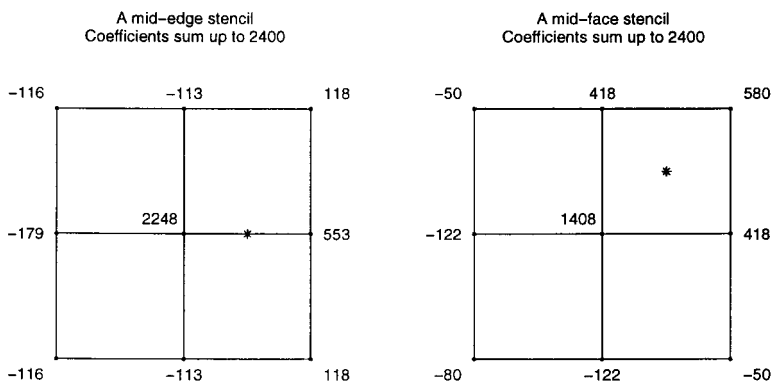
FIG. 5. Stencils for the modified Catmull–Clark scheme. A stencil for the edges emanating from the interpolatory vertex, and a stencil for the faces that share the interpolatory vertex. The interpolatory vertex is the vertex at the middle.

generates the tensor product cubic B-splines. This time, two stencils need to be calculated: One, for the edges emanating from the interpolatory vertex, and the other, for the faces near the interpolatory vertex. This time, we require our schemes to reproduce the polynomials $1, x, y, x^2 - y^2, xy$ that are reproduced by S. We use the remaining degrees of freedom to minimize the fairness measure (6.6), and then round the results to rational numbers. The resulting stencils are depicted in Fig. 5.

7. SUBDIVISION SCHEMES INTERPOLATING NORMAL VECTORS

In this section, we use the notion of finitely non-uniform subdivision schemes for interpolation of points with given normal vectors. We construct subdivision schemes that are uniform away from the origin, whose limit surfaces interpolate a given point, and have a prescribed normal vector at the origin of the parametric domain.

The basic idea is demonstrated by the following example: Let $\tilde{S}$ and $\tilde{S}_0$ denote two finitely non-uniform bivariate subdivision schemes that coincide with a uniform subdivision scheme S away from the boundary. Assume that both schemes generate C^1 limit functions, interpolating the control point at the origin, and that $\tilde{S}_0$ has the property mentioned in Corollary 4, namely that its limit functions have zero gradient at the origin. Given three dimensional control points $P = P^0: \mathbb{Z}^2 \to \mathbb{R}^3$, we define a new finitely non-uniform subdivision scheme by

$$
\begin{aligned}
P_1^{n+1} &= \tilde{S}P_1^n, \\
P_2^{n+1} &= \tilde{S}P_2^n, \\
P_3^{n+1} &= \tilde{S}_0 P_3^n, \qquad n = 0, 1, ...,
\end{aligned}
\tag{52}
$$

where $P_i^n(\cdot)$ denotes the ith component of the three dimensional vector $P^n(\cdot)$. Note that $\tilde{S}_0$ operates on the third coordinate of P^n while $\tilde{S}$ operates on the first two coordinates. As a result, if the limit surface has a normal vector at the origin, it must be exactly $(0, 0, 1)$. In case a different normal vector is given, we can rotate the given control points, and then follow the same construction.

In the following, we extend the above construction for the s-dimensional case. Our finitely non-uniform subdivision schemes operate on $s+1$ dimensional control points $P: \mathbb{Z}^s \to \mathbb{R}^{s+1}$. We assume that we have two s-variate

finitely non-uniform subdivision schemes $\tilde{S}$ and $\tilde{S}_0$ that coincide away from the origin, which are known to generate C^1 limit functions that interpolate the control point at the origin. Moreover we assume that $\tilde{S}_0$ has the property mentioned in Corollary 4, namely that its limit functions always have zero gradient at the origin, and that $\tilde{S}$ satisfies the conditions in Corollary 3.

Given a unit vector $N \in \mathbb{R}^{s+1}$, which stands for the prescribed normal at the origin, the first stage in our construction consists of finding any orthogonal $(s+1) \times (s+1)$ matrix Q with N as the last row. Then we define a new $s+1$ dimensional subdivision scheme R by

$$R = Q^t \begin{pmatrix} \tilde{S} & 0 & \cdots & 0 & 0 \\ 0 & \tilde{S} & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & \tilde{S} & 0 \\ 0 & 0 & \cdots & 0 & \tilde{S}_0 \end{pmatrix} Q. \tag{53}$$

In the above notation, $\tilde{S}$ and $\tilde{S}_0$ are elements in a matrix of subdivision operators. Thus the operation of the scheme R is represented by a $(s+1) \times (s+1)$ matrix of subdivision operators, and it operates on $(s+1)$-dimensional control points.

The geometric interpretation of R is the following: First we rotate the control points, so that the vector N is mapped to the vector $(0, 0, ..., 0, 1)$, then we apply $\tilde{S}$ on every coordinate, except for the last one where $\tilde{S}_0$ is applied. Finally, we rotate the control points back to their original coordinate system.

It is easy to see that R is interpolatory at the origin, since both $\tilde{S}$ and $\tilde{S}_0$ are interpolatory at the origin.

The regularity of the limit surfaces, for almost every set of initial control points, follows from Corollary 3. It follows from the construction that the last coordinate of any partial derivative of $QR^\infty P$ is zero at the origin, therefore the partial derivatives of $R^\infty P$ at the origin are all perpendicular to N. It is also simple to show that R is invariant to the specific choice of Q and to inverting the sign of the vector N, since two $(s+1) \times (s+1)$ orthogonal matrices Q_1, Q_2 that share the same last row (up to a sign change), satisfy the relation

$$Q_2 = \begin{pmatrix} H & 0 \\ 0 & \pm 1 \end{pmatrix} Q_1, \tag{54}$$

where H is an $s \times s$ orthogonal matrix. But

$$
\begin{pmatrix} H^{-1} & 0 \\ 0 & \pm 1 \end{pmatrix}
\begin{pmatrix}
\tilde{S} & 0 & \cdots & 0 & 0 \\
0 & \tilde{S} & & 0 & 0 \\
\vdots & & \ddots & & \vdots \\
0 & 0 & & \tilde{S} & 0 \\
0 & 0 & \cdots & 0 & \tilde{S}_0
\end{pmatrix}
\begin{pmatrix} H & 0 \\ 0 & \pm 1 \end{pmatrix}
$$

$$
=
\begin{pmatrix}
\tilde{S} & 0 & \cdots & 0 & 0 \\
0 & \tilde{S} & & 0 & 0 \\
\vdots & & \ddots & & \vdots \\
0 & 0 & & \tilde{S} & 0 \\
0 & 0 & \cdots & 0 & \tilde{S}_0
\end{pmatrix}.
$$

Another important observation is the following: Since $\tilde{S}$ and $\tilde{S}_0$ coincide away from the origin with a subdivision scheme S, then R coincides away from the origin with the diagonal subdivision scheme

$$
\begin{pmatrix}
S & 0 & \cdots & 0 & 0 \\
0 & S & & 0 & 0 \\
\vdots & & \ddots & & \vdots \\
0 & 0 & & S & 0 \\
0 & 0 & \cdots & 0 & S
\end{pmatrix},
$$

since $Q^t I Q = I$, where I is the $(s+1) \times (s+1)$ identity matrix. The importance of that observation, is that outside a finite neighborhood of the origin, the subdivision schemes that we construct are invariant to the choice of N. The effect of the special normal vector interpolation condition therefore remains local.

It follows from our construction that the scheme R is invariant to orthogonal transformations, in the following sense: If we transform the initial control points P as well as the given normal vector N by left multiplication by an orthogonal $(s+1) \times (s+1)$ matrix A, and then calculate Q and apply the scheme R, the resulting control points are the same as if we first apply the scheme R, and then apply A.

One application of such schemes is to prescribe local and global minima and maxima of the limit surface, with respect to a given direction. This

leads to the following questions: How should we choose the scheme S_0 in order to prescribe local extremes of the limit surface? What are the conditions on P that guarantee a local/global extremum? We will answer this questions in the restricted case where the uniform scheme away from the origin, S, has positive stencil coefficients (which is the case for B-spline and box spline subdivision schemes, for example), which yields

$$\Phi(x) \geqslant 0, \qquad \forall x \in \mathbb{R}^s. \tag{56}$$

An analysis of shape preserving properties of subdivision schemes with positive coefficients is given in [10].

It is easy to see that the problem of prescribing an extremum of the limit surface of R at the origin, with respect to a given direction, is equivalent to the problem of requiring the limit function generated by $\tilde{S}_0$ to have a local/global minimum at the origin.

Let $P: \mathbb{Z}^s \to \mathbb{R}$ denote initial scalar control points such that $P(0) = 0$. We seek conditions on $\tilde{S}_0$ and on P, which guarantee that

$$\tilde{S}_0^\infty P(x) \geqslant 0, \qquad \forall x \in \mathbb{R}^s. \tag{57}$$

We define $\tilde{S}_0$ by

$$\tilde{S}_0 Q(\alpha) = \begin{cases} M_0 Q_{|T}(\alpha), & \alpha \in T, \\ S Q(\alpha), & \alpha \notin T, \end{cases} \quad \forall Q \in l(\mathbb{Z}^s), \tag{58}$$

and denote

$$P^0 = P, \qquad P^{n+1} = \tilde{S}_0 P^n, \qquad n > 0. \tag{59}$$

As shown in Section 5 there exists an open and bounded set $\Theta \subset \mathbb{R}^s$ that contains the origin, with the property that

$$x \in 2^{-n+1}\Theta \backslash 2^{-n}\Theta \Rightarrow \tilde{S}_0^\infty Q(x)$$
$$= \sum_{\alpha \in T \backslash \{0\}} Q^n(\alpha)\, \Phi(2^n x - \alpha), \qquad \forall n \geqslant 0, \quad \forall Q \in l(\mathbb{Z}^s), \tag{60}$$

and

$$x \in \mathbb{R}^s \backslash 2\Theta \Rightarrow \tilde{S}_0^\infty Q(x) = S^\infty Q(x). \tag{61}$$

The following theorem provides a sufficient condition for the existence of a global minimum of the limit function $\tilde{S}_0^\infty P$ at the origin.

THEOREM 6.   *Under* (56), *if* $P^0_{|\mathbb{Z}^s \setminus \{0\}} > 0$ *and* $P^n_{|T \setminus \{0\}} > 0$ *for all* $n \geq 0$, *then* $\tilde{S}^\infty_0 P$ *has a strict global minimum at the origin.*

Proof.   Since $\tilde{S}_0$ is interpolatory at the origin we have that $P^n(0) = 0$ for all $n \geq 0$, therefore $\tilde{S}^\infty_0 P(0) = 0$. From (60) and (61) it follows that $\tilde{S}^\infty_0 P_{|\mathbb{R}^s \setminus \{0\}} > 0$.   ∎

The main condition of this theorem concerns the positivity of $P^n_{|T \setminus \{0\}}$, which is non-trivial since the scheme $\tilde{S}_0$ does not necessarily have positive coefficients. However the analysis is simpler if we only want a local minimum at the origin: Since $P^n_{|T} = M^n_0 P_{|T}$, as n tends to infinity the significant part of the vectors $P^n_{|T}$ consists of the eigenvectors of M_0 that correspond to the leading eigenvalues (excluding the eigenvalue 1). Therefore, a key requirement in the construction of the scheme $\tilde{S}_0$, is the existence of a leading eigenvector that is strictly positive on $T \setminus \{0\}$. This ensures the existence of a subset $L \subset l_0(T)$, of positive measure such that $\tilde{S}^\infty_0 P$ has a strict local minimum at the origin whenever $P_{|T} \in L$.

In the following, we construct such a scheme $\tilde{S}_0$ in the bivariate setting. We seek a scheme that is suitable for triangular subdivision, and coincides with Loop's scheme away from the origin. Taking the same steps as in Section 6, we set up the following constraints on the stencil of $\tilde{S}_0$ near the origin:

1.   The weights must sum up to 1.

2.   The weight of the value at the origin should be 34.

3.   The leading eigenvalue of the corresponding matrix M_0 (excluding the eigenvalue 1) should be 14.
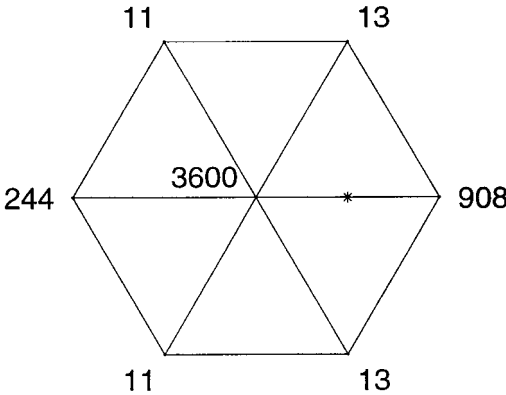
FIG. 6. A stencil calculated by minimizing a fairness measure, with zero gradient at the origin. The coefficients sum up to 4800.

Conditions 1 and 2 guarantee that the sum of all the weights excluding the weight at the origin, is 14, meaning that M_0 has an eigenvector (corresponding to the eigenvalue 14), which has the value 0 at the origin, and 1 at the points $(1, 0)$, $(1, 1)$, $(0, 1)$, $(-1, 0)$, $(-1, -1)$, $(0, -1)$. This suggests the existence of a combinations of the leading eigenvectors that is strictly positive on $T \backslash \{0\}$, which is important if we want to prescribe a local extremum at the origin. Condition 3 suggests that according to Theorem 5, $\tilde{S}_0$ will generate limit functions that are C^1 with bounded second derivatives.

Under the above constraints, we minimize the fairness measure (51), and once again, round the resulting weights to rational values. Figure 6 depicts the resulting stencil for $\tilde{S}_0$.

Note that the weights in this stencil are positive, and therefore the entire scheme $\tilde{S}_0$ is a non-negative scheme. In particular, it follows that the condition for a global minimum at the origin is very simple:

LEMMA 7. *Let $\tilde{S}_0$ denote a finitely non-uniform interpolating scheme which is convergent and non-negative. If $P(\alpha) > P(0)$ for all $\alpha \in \mathbb{Z}^s \backslash \{0\}$, then $\tilde{S}_0^\infty P$ has a strict global minimum at the origin.*

8. IMPLEMENTATION

In this section, we demonstrate surfaces generated by locally interpolating schemes, from initial triangular meshes.

In the first example, we prescribe a single point-interpolation conditions and two normal-vector interpolation conditions. We use Loop's scheme [8] away from the interpolation points, and near the interpolation points, we use the stencils shown in Fig. 4. Near interpolation points where a normal vector is given, we use the stencils shown in Figs. 4 and 6, combined as described in Section 7. Figure 7 shows the resulting surfaces.

In the second example, we prescribe two point-interpolation conditions and a single normal-vector interpolation condition, and apply a similar scheme. Figure 8 shows the resulting surfaces.

In the third example, we design surfaces that have a single smooth boundary, and a prescribed extremum point, with a prescribed normal vector. We use Loop's scheme away from the extremum point, and away from the boundary. Near the boundary we use the schemes developed in [7], and near the extremum point, we use the stencils shown in Figs. 4 and 6, combined as described in Section 7. Figure 9 shows the resulting surfaces. The prescribed normal vector is shown by a black line segment.
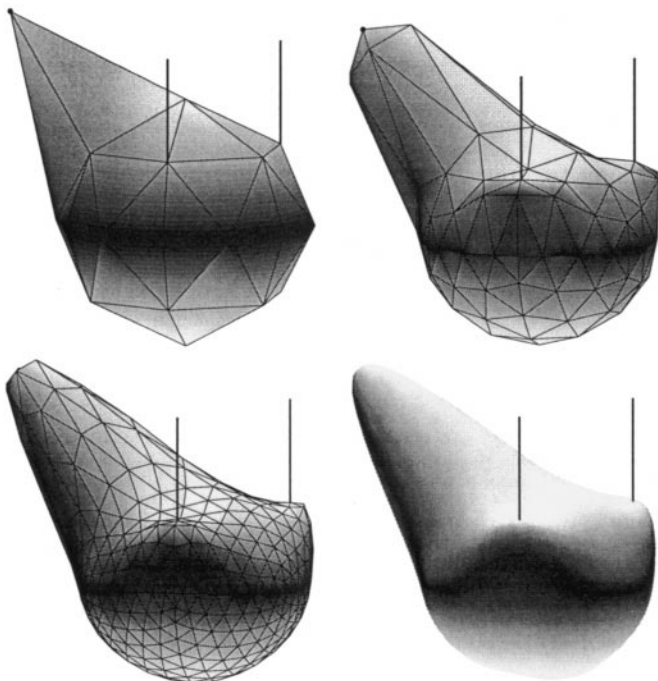
FIG. 7. Three iterations of a finitely-non uniform scheme. Prescribed normal vectors are shown as black lines. The interpolation point is marked by a black bold point.
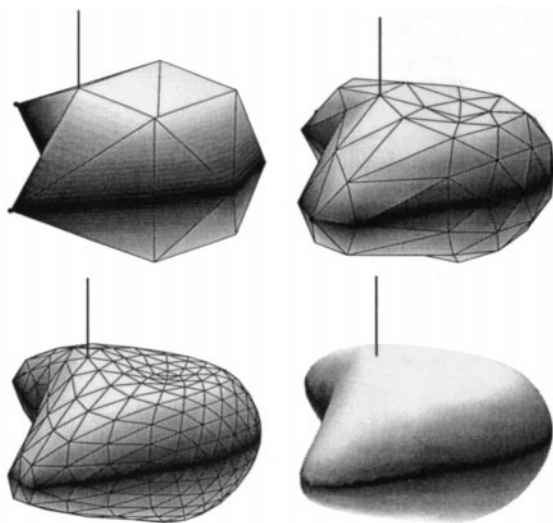
FIG. 8. Three iterations of a finitely-non uniform scheme. The prescribed normal vectors is shown as a black line. The interpolation point are marked by black bold points.
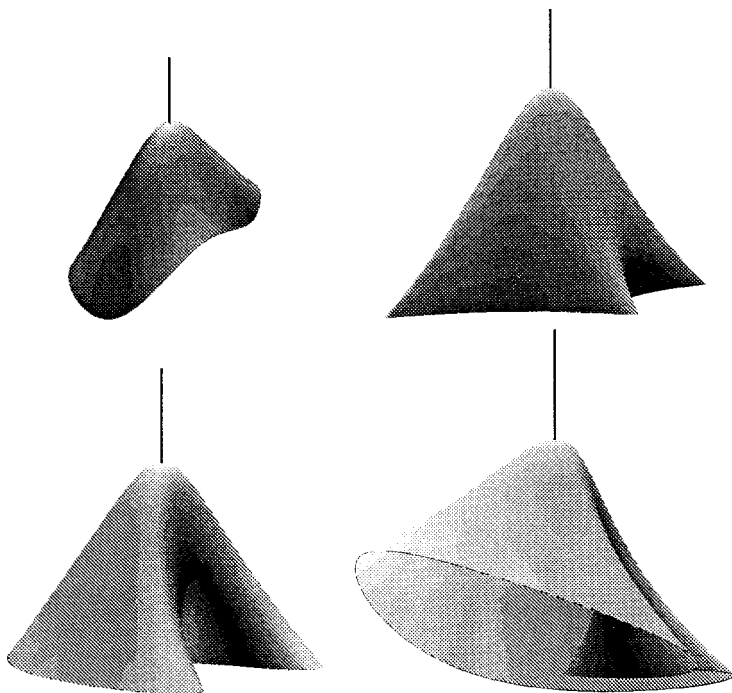
FIG. 9. Four limit surfaces with a constrained boundary and a prescribed normal vector at a single point.

ACKNOWLEDGMENTS

REFERENCES

1. A. S. Cavaretta, W. Dahmen, and C. A. Micchelli, Stationary subdivision, *Mem. Amer. Math. Soc.* **453** (1991).
2. C. de Boor, K. Höllig, and S. Riemenschneider, "Box Splines," Applied Mathematical Sciences, Vol. 98, Springer-Verlag, New York/Berlin, 1993.
3. N. Dyn, Subdivision schemes in computer aided geometric design, *in* "Advances in Numerical Analysis. II. Subdivision Algrorithms and Rational Functions" (W. A. Light, Ed.), pp. 36–104, Oxford Univ. Press, London, 1992.
4. N. Dyn, J. A. Greogory, and D. Levin, A butterfly subdivision scheme for surface interpolation with tension control, *ACM Trans. Graphics* **9** (1990), 160–169.

5. N. Dyn, J. A. Greogory, and D. Levin, Analysis of uniform binary subdivision schemes for curve design, *Const. Approx.* **7** (1991), 127–147.

6. N. Dyn and D. Levin, Interpolatory subdivision schemes for the generation of curves and surfaces, *in* "Multivariate Approximation and Interpolation" (W. Haussmann and K. Jetter, Eds.), pp. 91–106, Birkhäuser Verlag, Basel, 1990.

7. A. Levin, Analysis of combined subdivision schemes for the design of surfaces satisfying boundary conditions, http://www.math.tau.ac.il/~adilev, 1998.

8. C. Loop, "Smooth Spline Surfaces Based on Triangles," Master's thesis, University of Utah, Department of Mathematics, 1987.

9. D. Sherman Riemenschneider and Zuowei Shen, Multidimensional interpolatory subdivision schemes, analysis and approximation theory seminar, University of Alberta, 1996.

10. I. Yad-Shalom, "Geometric Properties of Curves ans Surfaces Generated by Translates of a Single Function," Ph.D. thesis, Tel-Aviv University, 1993.